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THE RELATIVE TRACE FORMULA FOR GROUPS WITH INVOLUTION

BY

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ABSTRACT

The Relative Trace Formula is a tool for establishing the Langlands functoriality principle. For a given reductive group G and involution θ on it we construct a new group G', formulate the Relative Trace Formula for groups G and G' and take some steps towards the proof of this formula.

1. Introduction

Let F be a number field and G a reductive simple group which is either split over F, or obtained by restriction of scalars from a split group G_1 over E, where $E \supset F$ is a quadratic extension of fields.

We define **a splitting field** of G to be F in the first case and E in the second case. In the first case $G_{\bar{F}} = G \otimes_F \bar{F}$ is a simple group, and in the second case $G_{\bar{F}} = G_{1\bar{F}} \times G_{1\bar{F}}$.

Let θ be an involution of G also defined over F. Then θ induces an involution on $G_{\bar{F}}$. If $G_{\bar{F}}$ is not simple, we consider those involutions θ that interchange two copies of $G_{1\bar{F}}$, i.e., $\theta(G_{1\bar{F}} \times \{e\}) = \{e\} \times G_{1\bar{F}}$. This actually means that θ involves a Galois action and we call this θ **Galois**. If θ is Galois then the Galois group $\mathrm{Gal}(E/F)$ of a splitting field E over F acts on the group G. We denote the action of the nontrivial element of $\mathrm{Gal}(E/F)$ by an overbar.

We assume that there exists a pair (B,T) of Borel subgroup and torus of G both defined over F which is θ-invariant.

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One defines $S = \{x \in G : x\theta(x) = 1\}$. Then the group G acts on S by $g.s = gs\theta(g)^{-1}$. We denote by $\{\varepsilon\}$ the system of representatives of orbits, by S_ε the orbit corresponding to ε, and by H_ε the stabilizer of ε. Obviously $S_\varepsilon = G/H_\varepsilon$. Denote by H the group of fixed points of involution θ.

By $G_{\mathbb{A}}$ we denote the group G over the adeles.

First of all let us give an important definition of a **distinguished** representation which we shall need in order to formulate the main conjecture.

Definition: A cuspidal representation $(\pi, G_{\mathbb{A}}, V_\pi)$ is called distinguished with respect to the subgroup R of G if there exists $\varphi \in V_\pi$ such that

$$\int_{R_F \setminus R_{\mathbb{A}}} \varphi(h)\, dh \neq 0.$$

CONJECTURE: *Automorphic generic cuspidal representations of G distinguished with respect to one of H_ε can be characterized as those that come from automorphic generic cuspidal representations of some reductive group G' under the map induced by the homomorphism ${}^LG' \mapsto {}^LG$ according to the Langlands functoriality principle.*

One way to prove this conjecture is to use the Relative Trace Formula of Jacquet. This is an identity of the form

$$\int_{N'_F \setminus N'_{\mathbb{A}}} \int_{N'_F \setminus N'_{\mathbb{A}}} K_{f'}(n_1, n_2)\psi'(n_1{}^{-1}n_2)\, dn_1\, dn_2 = \sum_{\varepsilon} \int_{N_F \setminus N_{\mathbb{A}}} \int_{H_{\varepsilon F} \setminus H_{\varepsilon \mathbb{A}}} K_{f_\varepsilon}(n, h_\varepsilon)\psi(n)\, dn\, dh_\varepsilon.$$

Here, N and N' are the maximal unipotent subgroups of G and G' respectively, ψ and ψ' are nondegenerate characters of N and N' respectively, and K_{f_ε} and $K_{f'}$ are the kernels of the operators corresponding to the functions f_ε and f' which are smooth and of compact support on $G_{\mathbb{A}}$ and $G'_{\mathbb{A}}$ respectively. The proof of the Relative Trace Formula consists of three steps.

STEP 1: To represent each side of the identity as a sum of orbital integrals; namely, to rewrite it in the form

$$\sum_{\gamma' \in N' \backslash G'/N'} I_{\gamma'}(f', \psi') = \sum_{\varepsilon} \sum_{\gamma \in N \backslash G/H_\varepsilon} I_\gamma(f_\varepsilon, \psi).$$

Definition: The double coset γ is called **ψ-admissible** if the corresponding orbital integral $I_\gamma(*, \psi)$ does not vanish identically.

STEP 2: To match the admissible double cosets $\gamma' \leftrightarrow \gamma$ on both sides in a natural way.

Every admissible integral $I_\gamma(f_\varepsilon, \psi)$ and $I'_\gamma(f', \psi')$ is a product of local integrals, as will be shown below. Hence it remains to prove that certain local integrals are equal.

STEP 3: *Fundamental Lemma.* For every f'_ν there is f_{ε_ν} and vice versa such that

$$I_{\gamma_\nu}(f_{\varepsilon_\nu}, \psi) = I'_{\gamma_\nu}(f'_\nu, \psi').$$

Moreover, this local matching is compatible with the map between the Hecke algebras of G' and G. This map is dual to the homomorphism of L-groups

$$r\colon {}^L G'_\nu \mapsto {}^L G_\nu.$$

In particular, if ν is outside of the finite "bad" set of primes and f'_ν belongs to the Hecke algebra of G'_ν, the f_{ε_ν} in the Hecke algebra of G defined by

$$({f'}_\nu)^S(t) = (f_{\varepsilon_\nu})^S(r(t))$$

will be a matching function to f'_ν. Here f^S denotes the Satake transform of the function f. The identity is well-defined since for almost all ν the representative ε_ν belongs to the orbit of 1.

Step 3 is usually the most difficult step.

In the present work we define the group G' and do steps one and two. The fundamental lemma is formulated explicitly.

THEOREM: *Under certain conditions on θ the admissible cosets of the group G match the admissible cosets of the group G', where G'_F is a subgroup of G_F defined by*

$$G' = \{g\colon \theta^{D_\psi}(g) = n_{w_\Delta}\, {}^t g^{-1}\, {n_{w_\Delta}}^{-1}\}.$$

Here, n_{w_Δ} is a canonical representative of the longest element of the Weyl group of G and θ^{D_ψ} is the involution θ conjugated by an element D_ψ of the torus depending on θ and ψ.

Remark: Below we define for every θ and ψ an element $z_{\theta,\psi}$ of the center Z_G. The condition in the theorem is that this $z_{\theta,\psi} = 1$. Moreover, we show that for an involution not involving the Galois action this condition is equivalent to the existence of admissible cosets.

2. Examples

2.1. Let $E \supset F$ be a quadratic extension of the number field and $G = \mathrm{GL}_n(E)$ is the group obtained by the restriction of scalars from the group $\mathrm{GL}_n(F)$. The involution θ on G is defined by $\theta(g) = \bar{g}$. Then $G' = U_n(E/F)$ is a unitary group and $H = \mathrm{GL}_n(F)$. The corresponding map of L-groups is defined by

$$r\colon \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F) \mapsto \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$$
$$(g,\sigma) \mapsto (g, {}^tg^{-1}, \sigma).$$

This case is discussed in [F].

2.2. Let G be as above and the involution be defined by

$$\theta(g) = n_{w_\Delta}\, {}^t\bar{g}^{-1}\, n_{w_\Delta}{}^{-1},$$

where n_{w_Δ} is as in the theorem. Then $H = U_n(E/F)$ and $G' = \mathrm{GL}_n(F)$. This is the Base Change case, dual to the case 2.1. The corresponding map of L-groups is defined by

$$r\colon \mathrm{GL}_n(\mathbb{C}) \times \mathrm{Gal}(E/F) \mapsto \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$$
$$(g,\sigma) \mapsto (g, g, \sigma).$$

In this case the Relative Trace Formula and the conjecture was proved for $n = 2, 3$ by Jacquet and Ye [JY1],[JY2], [J1]. See also [JLR].

2.3. Let G_E be the group obtained by the restriction of scalars from a split group G_F. The involution is defined by $\theta(g) = \bar{g}$. Then G' is either the group G_F or the quasi-split outer form of G_F according as -1 lies in the Weyl group of G_F or not. This case is extremely important because there are some ways for proving the fundamental lemma in this case.

2.4. Let $G = \mathrm{GL}_{2n}(F)$ and $\theta(g) = J\, {}^tg^{-1} J^{-1}$, the involution defining a symplectic group. In this case there are no admissible cosets, i.e., the contribution of the RHS of the Relative Trace Formula is identically zero. Considering the spectral side one concludes that there are no cuspidal generic representations of GL_{2n} distinguished with respect to symplectic group. Jacquet and Rallis in [JR] considered a slightly different Relative Trace Formula, namely, the character ψ was taken degenerate. Then there exist admissible cosets and they match the admissible cosets of $G' = \mathrm{GL}_n$. They also prove the fundamental lemma.

2.5. $G = \mathrm{GL}_n(F)$ and $\theta(g) = A\,{}^tg^{-1}A^{-1}$, the involution defining the orthogonal group. Then $H = \mathrm{SO}_n$ and $G'_F = \mathrm{GL}_n(F)$. In this case, G' over the adeles is conjecturally the double cover of GL_n and the result probably generalizes the work of Waldspurger. Namely, it is known that automorphic representations of GL_2 distinguished with respect to an orthogonal group correspond to automorphic representations of the double cover of GL_2. Moreover, an orthogonal group in this case can be taken to be split. See [J2], [J3].

3. Notations

3.1. Notations on the structure of the group. Throughout the paper all data related to the left hand side of the Relative Trace Formula will be marked with a prime. For the group G we denote by B the Borel subgroup of G which is θ-stable. Then $B = NT$, where N is the maximal unipotent radical of B and T is the maximal torus. We denote by $R(G,T)$ the set of roots with respect to the maximal torus T and by Δ the set of simple roots. Obviously θ acts on the set of roots and this action preserves the set of simple roots and the set of positive roots. Similarly, θ acts on the Weyl group of G. All induced actions we also denote by θ.

Let us fix the one parameter additive subgroups x_α of G for every root $\alpha \in R(G,T)$. This is an isomorphism of the additive group of the splitting field of G onto a closed subgroup U_α of G, normalized by T such that

$$tx_\alpha(\zeta)t^{-1} = x_\alpha(\alpha(t)\zeta).$$

For every $w \in W = N_G(T)/T$ one can choose its representative $n_w \in G$ such that for every $\alpha \in R(G,T)$ one has

$$n_w x_\alpha(\zeta) {n_w}^{-1} = x_{w.\alpha}(\zeta).$$

We shall always work with such representatives and call them **standard**.

Moreover, we define the action θ_α on the additive group of the splitting field of G for every simple root α by

$$\theta(x_\alpha(\zeta)) = x_{\theta.\alpha}(\theta_\alpha(\zeta)).$$

These actions define additive automorphisms of the splitting field.

By transpose we shall always mean the antiautomorphism of the group defined by ${}^tx_\alpha(\zeta) = x_{-\alpha}(\zeta)$.

3.2. THE CHARACTERS ψ AND ψ'. There is a connection between the characters ψ' and ψ. The character ψ' is nondegenerate. That means

$$\psi'\left(\prod_{\alpha>0} x_\alpha(\zeta_\alpha)\right) = \tau\left(\sum_{\alpha \text{ is simple}} \zeta_\alpha\right)$$

where τ is a fixed additive character of F.

If θ does not involve Galois action then we define

$$\psi\left(\prod_{\alpha>0} x_\alpha(\zeta_\alpha)\right) = \tau\left(\sum_{\alpha \text{ is simple}} \zeta_\alpha\right).$$

If θ involves Galois action then we define two inequivalent characters ψ_+ and ψ_- on N as follows:

$$\psi_+\left(\prod_{\alpha>0} x_\alpha(\zeta_\alpha)\right) = \tau\left(\sum_{\alpha \text{ is simple}} \zeta_\alpha + \overline{\zeta_\alpha}\right),$$

and

$$\psi_-\left(\prod_{\alpha>0} x_\alpha(\zeta_\alpha)\right) = \tau\left(\sum_{\alpha \text{ is simple}} \frac{\zeta_\alpha - \overline{\zeta_\alpha}}{\zeta_0}\right),$$

where ζ_0 is a fixed element of the splitting field E such that $\overline{\zeta_0} = -\zeta_0$.

When the involution θ is Galois and we put the character ψ_+ or ψ_- on the right hand side, we refer to the admissible orbits of the right hand side, as ψ_+-admissible orbits or ψ_--admissible orbits respectively.

3.3. DEFINITION OF D_ψ AND $z_{\theta,\psi}$. The involution may or may not involve Galois action. In case it does we call the involution Galois and then $\theta_\alpha(\zeta) = a_\alpha\overline{\zeta}$, and if it does not then $\theta_\alpha(\zeta) = a_\alpha\zeta$ for some constants a_α.

If θ is not Galois then we define

$$D \in T\colon \alpha(D) = -1/a_{\theta.\alpha} \quad \forall\alpha \in \Delta.$$

Trivial computations show that the involution θ^D defined by

$$\theta^D(g) = D\theta(g)D^{-1}$$

satisfies $\theta^D_\alpha(\zeta) = -\zeta$.

If θ is Galois then we define

$$D_\pm \in T\colon \alpha(D) = \mp 1/a_{\theta.\alpha} \quad \forall\alpha \in \Delta.$$

Similarly, we define involutions $\theta^{D_\pm}$ that satisfy

$$\theta_\alpha^{D_\pm}(\zeta) = \mp\bar{\zeta},$$

for every simple root α. Next we define

$$D_\psi = D_\pm \quad \text{for } \psi = \psi_\pm$$

if θ is Galois and

$$D_\psi = D$$

if θ is not Galois. Note that such D_ψ satisfies $D_\psi\theta(D_\psi) \in Z_G$, hence θ^{D_ψ} is really an involution. We define for any θ

$$z_{\theta,\psi} = (D_\psi\theta(D_\psi))^{-1}.$$

Note that D_ψ is defined up to center, hence $z_{\theta,\psi} \in Z_G/\{z\theta(z)\}$.

3.4. OTHER. If α and β are roots such that $\alpha \neq -\beta$ then

$$[x_\beta(\zeta_\beta), x_\alpha(\zeta_\alpha)] = \prod_{i\beta+j\alpha \text{ is a root}} x_{\alpha+\beta}(c_{i\beta,j\alpha}{\zeta_\beta}^i{\zeta_\alpha}^j).$$

Here $[a,b]$ denotes the commutator $a^{-1}b^{-1}ab$. In the present work we use this expression only in the case where the unique linear combination of roots α and β which is a root is $\alpha+\beta$. In this case, considering the expression $\theta([x_\beta(\zeta_\beta), x_\alpha(\zeta_\alpha)])$ one sees that

$$c_{\beta,\alpha}\theta_\alpha(\zeta_\alpha)\theta_\beta(\zeta_\beta) = c_{\theta.\beta,\theta.\alpha}\theta_{\alpha+\beta}\zeta_\alpha\zeta_\beta).$$

We shall use this identity later.

4. Proof of Step 1

Let us write both sides as sums of orbital integrals. We have

$$\begin{aligned} LHS &= \int_{N'_F \setminus N'_{\mathbb{A}}}\int_{N'_F \setminus N'_{\mathbb{A}}} \sum_{\gamma' \in G'_F} f(n_1{}^{-1}\gamma' n_2)\psi'(n_1{}^{-1}n_2)\,dn_1\,dn_2 \\ &= \int_{N'_F \setminus N'_{\mathbb{A}}}\int_{N'_F \setminus N'_{\mathbb{A}}} \sum_{\substack{\gamma' \in N'_F\backslash G'_F/N'_F \\ (m_1,m_2)\in O_{\gamma'}{}_F\backslash N'_F\times N'_F}} f'((m_1n_1)^{-1}\gamma' m_2n_2) \\ &\qquad\qquad \psi'((m_1n_1)^{-1}m_2n_2)\,dn_1\,dn_2 \\ &= \sum_{\gamma' \in N'_F\backslash G'_F/N'_F} \iint_{O_{\gamma'}{}_F\backslash N'_{\mathbb{A}}\times N'_{\mathbb{A}}} f'(n_1{}^{-1}\gamma' n_2)\psi'(n_1{}^{-1}n_2)\,dn_1\,dn_2 \end{aligned}$$

where $O_{\gamma' F} = \{(n_1, n_2) \in N_F \times N_F \mid {n_1}^{-1}\gamma' n_2 = \gamma'\}$. Thus

$$LHS = \sum_{\gamma' \in N'_F \backslash G'_F / N'_F} \int_{O_{\gamma' \mathbf{A}} \backslash N'_{\mathbf{A}} \times N'_{\mathbf{A}}} f'({n_1}^{-1}\gamma' n_2)\psi'({n_1}^{-1}n_2)\, dn_1\, dn_2 \times \int_{O_{\gamma' F} \backslash O_{\gamma \mathbf{A}}} \psi'({v_1}^{-1}v_2)\, dv_1\, dv_2,$$

where $O_{\gamma' \mathbf{A}} = \{(n_1, n_2) \in N_{\mathbf{A}} \times N_{\mathbf{A}} \mid {n_1}^{-1}\gamma' n_2 = \gamma'\}$.

Similar computations for the right hand side show

$$RHS = \sum_{\epsilon} \sum_{\gamma \in N_F \backslash G_F / H_{\epsilon F}} \int_{O_{\gamma \mathbf{A}} \backslash N_{\mathbf{A}} \times H_{\epsilon \mathbf{A}}} f_\epsilon(n^{-1}\gamma h_\epsilon)\psi(n)\, dn\, dh_\epsilon \times \int_{O_{\gamma F} \backslash O_{\gamma \mathbf{A}}} \psi(n)\, dn,$$

where $O_{\gamma \mathbf{A}} = \{(n, h_\epsilon) \in N_{\mathbf{A}} \times H_{\epsilon \mathbf{A}} \mid n^{-1}\gamma h_\epsilon = \gamma\}$.

5. The definition of admissible cosets revisited

Now we can write down the equivalent definition of admissibility of orbits, which will be used in our construction.

Definition: The double coset on the left hand side represented by γ is called **admissible** if

$$\int_{O_{\gamma' F} \backslash O_{\gamma' \mathbf{A}}} \psi'({v_1}^{-1}v_2)\, dv_1\, dv_2 \neq 0.$$

Similarly, the double coset on the right hand side represented by γ is called **admissible** if

$$\int_{O_{\gamma F} \backslash O_{\gamma \mathbf{A}}} \psi(n)\, dn \neq 0.$$

This is obviously equivalent to the triviality of characters ψ and ψ' on the corresponding sets O_γ and $O_{\gamma'}$.

6. Description of the representative of orbits

6.1. LEFT HAND SIDE. By the Bruhat decomposition $G' = B'W'B'$, where B' is a Borel subgroup and $W' = \{n_w \mid w \in N'_G(T')/T'\}$ is the finite set of the fixed representatives of the Weyl group in G'. Hence $N' \backslash G'/N' = W'T'$, where $T' \subset G'$ is a maximal torus. Thus we may assume γ' to be of the form $n_{w'}t'$ with $w' \in W', n_{w'}t' \in G'$.

6.2. RIGHT HAND SIDE. According to the result of Springer the double cosets $N\backslash G/H_\varepsilon$ correspond via $x\varepsilon\theta(x)^{-1}$ to the orbits of N on S_ε. Springer in [S] proved that every such orbit intersects the normalizer of the torus and it is easy to see that the intersection is one point. In order to use the result of Springer one needs the existence of a θ-invariant pair (B,T) in G, hence the condition on θ is necessary.

Thus we may assume $\gamma \in N\backslash G/H_\varepsilon$ to be of the form $x^\varepsilon_{w,t}$, where

$$x^\varepsilon_{w,t}\,\varepsilon\,\theta(x^\varepsilon_{w,t})^{-1} = tn_w \quad \text{and} \quad tn_w\theta(tn_w) = 1.$$

Let us introduce some notations: for every $J \subset \Delta$ (the set of all simple roots) we denote by R_J the root system with basis J, M_J the corresponding Levi subgroup, W_J the corresponding Weyl group and by w_J the longest element in W_J. In particular, w_Δ is the longest element in W.

7. Description of the admissible orbits

7.1. LEFT HAND SIDE.

PROPOSITION 1: *The element $\gamma' = n_{w'}t'$ represents the admissible orbit in G' iff*
(a) $w' = w_{\Delta'}w_{J'}$, *for some $J' \subset \Delta'$,*
(b) $t' \in Z_{M_{J'}}$, *the center of $M_{J'}$.*

7.2. RIGHT HAND SIDE. We noted above that double cosets in $\bigcup_\varepsilon N\backslash G/H_\varepsilon$ are parametrized by elements of $S\cap N_G(T)$. We call an element $tn_w \in S$ **admissible** if the corresponding double coset is admissible.

PROPOSITION 2: *An element $tn_w \in N_G(T)$ is admissible iff*
(a) $tn_w\theta(tn_w) = 1$, *i.e.,* $tn_w \in S$,
(b) $w = w_\Delta w_J$ *and* $w\theta(w) = 1$, *for some $J \subset \Delta$,*
(c) $\tilde{t} \in Z_{M_{\theta(J)}}$, *the center of $M_{\theta(J)}$, where* $\tilde{t} = tn_wD_\psi{}^{-1}n_w{}^{-1}$,
(d) $\theta^{D_\psi}(g) = z_{\theta,\psi}\, n_{w_\Delta}{}^t g^{-1} n_{w_\Delta}{}^{-1}$, *where* $g = n_w\theta(\tilde{t})$.

Note that if $z_{\theta,\psi} = 1$, the element g in Proposition 2, (d) belongs to the group G' defined in the Theorem.

8. Proof of Proposition 1

As before

$$\begin{aligned} O_{\gamma'_{\mathbf{A}}} &= \{(n_1,n_2) \in N'_{\mathbf{A}} \times N'_{\mathbf{A}} \mid n_1{}^{-1}n_{w'}t'n_2 = n_{w'}t'\} \\ &= \{(n_1,n_2) \in N'_A \times N'_{\mathbf{A}} \mid n_1 = n_{w'}t'n_2t'^{-1}n_{w'}{}^{-1}\}. \end{aligned}$$

Thus $n_{w'}t'n_2t'^{-1}n_{w'}^{-1}$ and $n_2 \in N'_{\mathbb{A}}$, hence $n_2 \in N'_{\mathbb{A}} \cap n_{w'}^{-1}N'_{\mathbb{A}}n_{w'}$. So

$$\begin{aligned}
&\int_{O_{\gamma'_F}\backslash O_{\gamma'_{\mathbb{A}}}} \psi'(v_1^{-1}v_2)\,dv_1\,dv_2\\
&= \int_{N'_{\mathbb{A}}\cap n_{w'}^{-1}N'_{\mathbb{A}}n_{w'}} \psi'(n_{w'}t'n^{-1}t'^{-1}n_{w'}^{-1}n)\,dn\\
&= \prod_{\substack{\alpha'>0\\ w'.\alpha'>0}} \int_{F\backslash\mathbb{A}} \psi'(n_{w'}t'x_{\alpha'}(-\zeta_{\alpha'})t'^{-1}n_{w'}^{-1}x_{\alpha'}(\zeta_{\alpha'}))\,d\zeta_{\alpha'}\\
&= \prod_{\substack{\alpha'>0\\ w'.\alpha'>0}} \int_{F\backslash\mathbb{A}} \psi'(x_{w'.\alpha'}(-\alpha'(t)\zeta_{\alpha'})x_{\alpha'}(\zeta_{\alpha'}))\,d\zeta_{\alpha'}.
\end{aligned}$$

We then see that for every root α' such that $w'.\alpha' > 0$ one has $w'.\alpha'$ also is simple. Note that this exactly means that $w' = w_{\Delta'}w_{J'}$ where J' is a subset of simple roots α such that $w'.\alpha'$ is simple. Indeed, any element w of the Weyl group is uniquely determined by the set of roots that w send to negative ones. Obviously the element $w_{\Delta'}w_{J'}$ sends any root α which is not in J' to a negative root and any root α in J' to a simple root. Moreover, for every $\alpha' \in J'$ one has $\alpha'(t') = 1$, i.e., $t' \in Z_{M_{J'}}$. The proposition follows. ∎

9. Proof of Proposition 2

Recall that admissibility of γ representing the coset means that if $(n, h_\epsilon) \in O_\gamma$ then $\psi(n) = 1$. One has

$$(n, h_\epsilon) \in O_\gamma \Leftrightarrow \gamma^{-1}n\gamma \in H_\epsilon \Leftrightarrow \gamma\epsilon\theta(\gamma)^{-1}\theta(n)(\gamma\epsilon\theta(\gamma)^{-1})^{-1} = n.$$

For $\gamma = x^\epsilon_{w,t}$ one has

$$\gamma\epsilon\theta(\gamma)^{-1} = tn_w.$$

So admissibility means

$$tn_w\theta(n)n_w^{-1}t^{-1} = n \Rightarrow \psi(n) = 1.$$

LEMMA 2.1: *If $tn_w\theta(tn_w) = 1$ and α is a simple root such that $\beta = w.\theta.\alpha > 0$, then for every $\zeta_\alpha \in F$ the element*

$$n = x_\alpha(\zeta_\alpha)\,x_\beta(\zeta_\beta)\,x_{\alpha+\beta}(\zeta),$$

where $\zeta_\beta = \beta(t)\theta_\alpha(\zeta_\alpha)$ and $\zeta = \frac{1}{2}c_{\beta,\alpha}\,\zeta_\alpha\zeta_\beta$, satisfies

$$tn_w\theta(n)n_w^{-1}t^{-1} = n.$$

Remark: If $\alpha+\beta$ is not a root we put $x_{\alpha+\beta}(\zeta)\equiv 1$.

Now let us prove the proposition assuming Lemma 2.1. (a) It is obvious since $tn_w \in S$ by our choice. (b) Assume that there is a simple root α such that $w.\theta.\alpha > 0$ and is not simple. Then considering n as in Lemma 2.1 one sees that the admissibility condition does not hold. Hence, for every α simple we have either $w.\alpha$ is also simple or $w.\alpha < 0$. This means that w is of the requested form. (c), (d) Let us take $\alpha \in \theta(J)$, i.e., $w.\theta.\alpha$ is a simple root, and consider $n = x_\alpha(\zeta_\alpha)x_\beta(\zeta_\beta)x_{\alpha+\beta}(\zeta)$ as in Lemma 2.1. Then

$$\alpha(t)\theta_\beta(\zeta_\beta)=\zeta_\alpha,\quad \beta(t)\theta_\alpha(\zeta_\alpha)=\zeta_\beta.$$

Assume that θ involves Galois action. In this case, if γ is ψ_+-admissible then for n as above one has $\psi_+(n)=1 \Leftrightarrow \zeta_\alpha+\zeta_\beta+\overline{\zeta_\alpha}+\overline{\zeta_\beta}=0$. This means

$$\alpha(t)a_\beta\overline{\zeta_\beta}+\zeta_\beta+\overline{\alpha(t)a_\beta}\zeta_\beta+\overline{\zeta_\beta}=0 \Leftrightarrow$$
$$(\alpha(t)a_\beta+1)\overline{\zeta_\beta}=-(\overline{\alpha(t)a_\beta}+1)\zeta_\beta.$$

Since $\overline{\zeta}/\zeta \neq$ const we have $\alpha(t)=-1/a_\beta \ \forall\alpha\in\theta(J)$.

Similarly, if γ is ψ_--admissible, then we have $\alpha(t)=1/a_\beta \ \forall\alpha\in\theta(J)$.

Now assume that θ does not involve Galois action. In this case, if γ is admissible then for n as above one has $\psi(n)=1\Leftrightarrow\zeta_\alpha+\zeta_\beta=0$. Hence $\alpha(t)=-1/a_\beta \ \forall\,\alpha\in\theta(J)$.

Summing all the above we conclude that if tn_w is ψ-admissible then for such a case one has

$$tn_w\theta(tn_w)=1\Leftrightarrow \tilde{t}n_w\theta^{D_\psi}(\tilde{t}n_w)=z_{\theta,\psi},$$

where $\tilde{t}=tn_wD_\psi{}^{-1}n_w{}^{-1}$ and θ^{D_ψ} is a new involution defined by

$$\theta^{D_\psi}(g)=D_\psi\theta(g)D_\psi{}^{-1}.$$

Thus

$$\begin{aligned}\alpha(\tilde{t})&=\alpha(tn_wD_\psi{}^{-1}n_w{}^{-1})=(-1/a_{w.\theta.\alpha})(w^{-1}.\alpha(D_\psi{}^{-1}))\\&=(-1/a_{w.\theta.\alpha})(-a_{w.\theta.\alpha})=1\quad\forall\alpha\in\theta(J).\end{aligned}$$

This means that $\theta^{D_\psi}(\tilde{t})$ belongs to the center of M_J, in particular it commutes with n_{w_J}. One has

$$\begin{aligned}\tilde{t}n_w\theta^{D_\psi}(\tilde{t}n_w)=z_{\theta,\psi}\Leftrightarrow \tilde{t}n_w\theta^{D_\psi}(\tilde{t})n_w{}^{-1}&=z_{\theta,\psi}\theta^{D_\psi}(n_w)^{-1}n_w{}^{-1}\Leftrightarrow\\ \tilde{t}n_{w_\Delta}n_{w_J}\theta^{D_\psi}(\tilde{t})n_{w_J}{}^{-1}n_{w_\Delta}{}^{-1}&=z_{\theta,\psi}\theta^{D_\psi}(n_w)^{-1}n_w{}^{-1}\Leftrightarrow\\ \tilde{t}n_{w_\Delta}\theta^{D_\psi}(\tilde{t})n_{w_\Delta}{}^{-1}&=z_{\theta,\psi}\theta(n_w)^{-1}n_w{}^{-1}\Leftrightarrow\\ \theta^{D_\psi}(n_w\theta(\tilde{t}))&=z_{\theta,\psi}n_w{}^{-1}n_{w_\Delta}\theta(\tilde{t})^{-1}n_{w_\Delta}{}^{-1}\Leftrightarrow\\ \theta^{D_\psi}(n_w\theta(\tilde{t}))&=z_{\theta,\psi}n_{w_\Delta}{}^{t}(n_w\theta(\tilde{t}))^{-1}n_{w_\Delta}{}^{-1}.\end{aligned}$$

The proposition follows. ∎

It remained to prove Lemma 2.1. One has

$$tn_w\theta(n){n_w}^{-1}t^{-1} = x_\beta(\beta(t)\theta_\alpha(\zeta_\alpha))\, x_\alpha(\alpha(t)\theta_\beta(\zeta_\beta))$$
$$x_{\alpha+\beta}((\alpha+\beta)(t)\frac{c_{\beta,\alpha}}{2}\theta_{\alpha+\beta}(\zeta_\alpha\zeta_\beta)).$$

CLAIM 1: *One has* $\zeta_\alpha = \alpha(t)\theta_\beta(\zeta_\beta) \Leftrightarrow \zeta_\beta = \beta(t)\theta_\alpha(\zeta_\alpha)$.

CLAIM 2: *One has* $\theta_{\alpha+\beta}(\zeta_\alpha\zeta_\beta) = -\theta_\alpha(\zeta_\alpha)\,\theta_\beta(\zeta_\beta)$.

Assume that the claims are proved. Then

$$\begin{aligned} tn_w\theta(n){n_w}^{-1}t^{-1} &= x_\beta(\zeta_\beta)\, x_\alpha(\zeta_\alpha)\, x_{\alpha+\beta}\Big(-\frac{c_{\beta,\alpha}}{2}(\alpha+\beta)(t)\theta_\alpha(\zeta_\alpha)\theta_\beta(\zeta_\beta)\Big) \\ &= x_\beta(\zeta_\beta)\, x_\alpha(\zeta_\alpha)\, x_{\alpha+\beta}\Big(-\frac{c_{\beta,\alpha}}{2}\zeta_\alpha\zeta_\beta\Big) \\ &= x_\alpha(\zeta_\alpha)\, x_\beta(\zeta_\beta)\, x_{\alpha+\beta}\Big(c_{\beta,\alpha}\zeta_\alpha\zeta_\beta - \frac{c_{\beta,\alpha}}{2}\zeta_\alpha\zeta_\beta\Big) = n \end{aligned}$$

as required.

Proof of Claim 1: Denote as before $\zeta_\beta = \beta(t)\theta_\alpha(\zeta_\alpha)$. Then

$$\begin{aligned} x_\alpha(\zeta_\alpha) &= tn_w\theta(tn_w)x_\alpha(\zeta_\alpha)\theta(tn_w)^{-1}(tn_w)^{-1} \\ &= tn_w\theta(x_\beta(\beta(t)\theta_\alpha(\zeta_\alpha))){n_w}^{-1}t^{-1} \\ &= x_\alpha(\alpha(t)\theta_\beta(\beta(t)\theta_\alpha(\zeta_\alpha))). \end{aligned}$$

Hence

$$\zeta_\alpha = \alpha(t)\theta_\beta(\beta(t)\theta_\alpha(\zeta_\alpha)) = \alpha(t)\theta_\beta(\zeta_\beta)$$

as required.

Proof of Claim 2: Since $tn_w\theta(tn_w) = 1$ one has

$$tn_w\theta(tn_w\theta(n){n_w}^{-1}t^{-1}){n_w}^{-1}t^{-1} = n.$$

We computed before

$$\begin{aligned} n_1 = tn_w\theta(n){n_w}^{-1}t^{-1} &= x_\beta(\zeta_\beta)\, x_\alpha(\zeta_\alpha)\, x_{\alpha+\beta}((\alpha+\beta)(t)\theta_{\alpha+\beta}(\zeta)) \\ &= x_\alpha(\zeta_\alpha)\, x_\beta(\zeta_\beta)\, x_{\alpha+\beta}(c_{\beta,\alpha}\zeta_\alpha\zeta_\beta + (\alpha+\beta)(t)\theta_{\alpha+\beta}(\zeta)). \end{aligned}$$

Now repeating the procedure one has

$$tn_w\theta(n_1){n_w}^{-1}t^{-1} =$$
$$x_\alpha(\zeta_\alpha)\, x_\beta(\zeta_\beta)\, x_{\alpha+\beta}(c_{\beta,\alpha}\zeta_\alpha\zeta_\beta + (\alpha+\beta)(t)\theta_{\alpha+\beta}(c_{\beta,\alpha}\zeta_\alpha\zeta_\beta + (\alpha+\beta)(t)\theta_{\alpha+\beta}(\zeta)).$$

Hence, since $tn_w\theta(n_1){n_w}^{-1}t^{-1} = n$, one has by comparing parameters of $x_{\alpha+\beta}$,

$$\begin{aligned}\zeta =& c_{\beta,\alpha}\zeta_\alpha\zeta_\beta + (\alpha+\beta)(t)\theta_{\alpha+\beta}(c_{\beta,\alpha}\zeta_\alpha\zeta_\beta + (\alpha+\beta)(t)\theta_{\alpha+\beta}(\zeta))\\ =& c_{\beta,\alpha}\zeta_\alpha\zeta_\beta\\ &+ c_{\theta.\beta,\theta.\alpha}(\alpha+\beta)(t)\theta_\alpha(\zeta_\alpha)\theta_\beta(\zeta_\beta) + (\alpha+\beta)(t)\theta_{\alpha+\beta}((\alpha+\beta)(t)\theta_{\alpha+\beta}(\zeta))\\ =& (c_{\beta,\alpha} + c_{\theta.\beta,\theta.\alpha})\zeta_\alpha\zeta_\beta + (\alpha+\beta)(t)\theta_{\alpha+\beta}((\alpha+\beta)(t)\theta_{\alpha+\beta}(\zeta)) = \zeta.\end{aligned}$$

Since this holds for any ζ one concludes that

$$c_{\beta,\alpha} = -c_{\theta.\beta,\theta.\alpha}.$$

Recall that we have seen in section 3 on notations the identity

$$c_{\beta,\alpha}\theta_\alpha(\zeta_\alpha)\theta_\beta(\zeta_\beta) = c_{\theta.\beta,\theta.\alpha}\theta_{\alpha+\beta}(\zeta_\alpha\zeta_\beta).$$

Hence $\theta_{\alpha+\beta}(\zeta_\alpha\zeta_\beta) = -\theta_\alpha(\zeta_\alpha)\,\theta_\beta(\zeta_\beta)$ as required and the lemma is proved.

10. The matching

Now we can state explicitly the theorem that was stated first in the introduction.

THEOREM: *If $z_{\theta,\psi} = 1$, there is a matching between admissible double cosets of G and G' and it is described by*

$$x^\epsilon_{w,t} \mapsto \gamma_{w',t'}$$

where

$$n_w\theta(\tilde{t}) \stackrel{\text{def}}{=} n_w\theta(tn_wD_\psi^{-1}{n_w}^{-1}) = n_{w'}t',$$

with notations as above.

Proof: First of all, let us describe the structure of the group G'. There is a map $\tau:\Delta\mapsto\Delta'$ defined by

(i) $x_{\tau(\alpha)}(\zeta) = x_\alpha(\zeta)$ if $\theta.\alpha = -w_\Delta.\alpha$,

(ii) $x_{\tau(\alpha)}(\zeta) = x_\alpha(\zeta)x_{-w_\Delta.\theta.\alpha}(\zeta)$ if $\alpha \neq -w_\Delta.\theta.\alpha$ and $x_\alpha(\zeta), x_{-w_\Delta.\theta.\alpha}(\zeta)$ commute,

(iii) $x_{\tau(\alpha)}(\zeta) = x_{\alpha-w_\Delta.\theta.\alpha}(\zeta)$ if $\alpha \neq -w_\Delta.\theta.\alpha$ and $x_\alpha(\zeta), x_{-w_\Delta.\theta.\alpha}(\zeta)$ do not commute.

In all three cases, $\zeta\in F$ if θ is not Galois and $\zeta\in F'\simeq F$ where F' is the additive subgroup of E.

The map τ is onto, i.e., each simple root of G' is expressed in terms of simple roots of G as above.

Consider the admissible coset of RHS represented by the element $x^{\epsilon}_{w,t}$. Then we know that $n_w\theta(\tilde{t}) \in G'$ hence $n_w\theta(\tilde{t}) = n_{w'}t'$, where $t' \in T_{G'}, w \in W'$.

CLAIM: *If $x^{\epsilon}_{w,t}$ is RHS admissible, in particular if $w = w_\Delta w_J$, then $n_{w'}t'$ is LHS admissible, $w' = w_{\Delta'}w_{\tau(J)}$.*

Proof of Claim: One must prove that $w'.\alpha'$ is simple for every root $\alpha' \in \tau(J)$ and negative otherwise. Moreover, $\alpha'(t') = 1$ for all $\alpha \in \tau(J)$. Assume that $\alpha' = \tau(\alpha)$. Then

$$(*) \qquad n_{w'}t'x_{\alpha'}(\zeta){t'}^{-1}{n_{w'}}^{-1} = x_{w'.\alpha'}(\alpha'(t')\zeta).$$

There are 3 different cases:

(i) $\alpha = -w_\Delta.\theta.\alpha$. In this case $x_{\alpha'}(\zeta) = x_\alpha(\zeta)$. Then

$$(*) = n_w\theta(\tilde{t})x_\alpha(\zeta){\theta(\tilde{t})}^{-1}{n_w}^{-1} = x_{w.\alpha}(\alpha(\theta(t))\zeta).$$

Hence one sees that for $\alpha' \in \tau(J)$ one has $\alpha \in J$, therefore $w'.\alpha'$ is simple and $\alpha(t') = 1$. For $\alpha \notin J$ one has $w'.\alpha < 0$.

(ii) $\alpha \neq -w_\Delta.\theta.\alpha$ and $x_\alpha(\zeta), x_{-w_\Delta.\theta.\alpha}(\zeta)$ commute. In this case also $x_{\alpha'}(\zeta) = x_\alpha(\zeta)x_{-w_\Delta.\theta.\alpha}(\zeta)$. Then

$$\begin{aligned}(*) &= n_w\theta(\tilde{t})x_\alpha(\zeta)x_{-w_\Delta.\theta.\alpha}(\zeta){\theta(\tilde{t})}^{-1}{n_w}^{-1} \\ &= x_{w.\alpha}(\alpha(\theta(t))\zeta)x_{-w.w_\Delta.\theta.\alpha}(-w_\Delta.\theta.\alpha(\theta(t))\zeta).\end{aligned}$$

Hence one sees that for $\alpha' \in \tau(J)$ one has $\alpha \in J$, therefore $w.\alpha$ is simple and $\alpha(\theta(t)) = 1$. Note that $-w.w_\Delta.\theta.\alpha = -w_\Delta.\theta.w.\alpha = -w_{\theta(J)}.\theta.\alpha$. Hence for $\alpha \in J$ one has $-w.w_\Delta.\theta.\alpha$ is also simple and the corresponding one parametric subgroups commute. Since the whole expression belongs to G' one has $w_\Delta.\theta.\alpha(\theta(\tilde{t})) = 1$. If $\alpha' \notin \tau(J)$ then $w'.\alpha'$ is negative.

(iii) $\alpha \neq -w_\Delta.\theta.\alpha$ and $x_\alpha(\zeta), x_{-w_\Delta.\theta.\alpha}(\zeta)$ do not commute. In this case $x_{\alpha'}(\zeta) = x_\alpha(\zeta)x_{-w_\Delta.\theta\alpha}(\zeta)$. The proof of the claim in this case is similar to the proof of (ii).

So every RHS admissible coset corresponds to some LHS-admissible coset. Similarly, every LHS-admissible coset has a representative of the form $n_{w'}t'$ and this element lies in the normalizer of the torus T_G. Hence it is of the form $n_w\theta(\tilde{t})$ for some $w \in W, \tilde{t} \in T_G$. Then the element $x_{w,t}$, where t is obtained from $\tilde{t}$ as before, is RHS admissible. The proof is analogous to the proof of the claim. Hence there is a matching of the admissible double cosets on both sides, as claimed. ∎

11. When does $z_{\theta,\psi} = 1$?

11.1. First of all, if the center of the group G is trivial, then $z_{\theta,\psi}$ is necessarily equal to 1. For example, for $G = \mathrm{SO}_{2n+1}$, G of type G_2.

11.2. THE BASE CHANGE CASE. This case has great importance, namely, the case when G_E is obtained by restriction of scalars from a split group G_F, $\theta(g) = n_{w_\Delta}{}^t\bar{g}^{-1}n_{w_\Delta}{}^{-1}$. Then $G' = G_F$. In this case one has $D_- = 1$ and hence $z_{\theta,\psi_-} = 1$.

11.3. THE DUAL CASE TO THE BASE CHANGE CASE. In this case G is as above and $\theta(g) = \bar{g}$. See example 2.3. Here one has $D_+ = 1$ and hence $z_{\theta,\psi_+} = 1$.

11.4. THE CASE θ IS NOT GALOIS. In this case we can completely describe the situation when G is a group of the classical type (A_n, B_n, C_n, D_n).

PROPOSITION 3: *Let G be a classical group and θ is not Galois. Assume there exists θ on G with $z_{\theta,\psi} \neq 1$. Then*

(a) $G = \mathrm{GL}_{2n}, \mathrm{SL}_{2n}, \mathrm{SO}_{2n}, \mathrm{Sp}_{2n}$.

(b) *There are no admissible orbits of G.*

The typical example is $G = \mathrm{GL}_{2n}$, $H = Sp_{2n}$. See example 2.4.

Proof: (a) To prove this case we show that for all other types of groups one has $z_{\theta,\psi} = 1$. We do it case by case. There are two possibilities:

(i) Assume θ acts trivially on the Dynkin diagram of G. Then we show that for $G = \mathrm{SL}_{2n+1}, \mathrm{SO}_{2n+1}$ and all similitude groups like $\mathrm{GL}_n, \mathrm{GSp}_{2n}$ and GSO_n, one has $z_{\theta,\psi} = 1$.

(ii) Assume θ acts nontrivially on the Dynkin diagram of G. Then we show that for $G = \mathrm{GL}_{2n+1}, \mathrm{SL}_{2n+1}, \mathrm{GSO}_{2n}$ and SO_{2n} one has $z_{\theta,\psi} = 1$.

Proof of (i): In this case

$$\theta^{D_\psi}(g) = EgE^{-1}, \quad \text{where } \alpha(E) = -1 \text{ for any } \alpha \in \Delta.$$

1. $G = \mathrm{SL}_{2n+1}$. Then $D_\psi = \mathrm{diag}(d_1, \ldots, d_{2n+1})$. One has

$$D_\psi \theta^{D_\psi}(D_\psi) = D_\psi^2 = z_{\theta,\psi}^{-1} = \mathrm{diag}(z, \ldots, z),$$

where $d_i^2 = z$ and $z^{2n+1} = 1$. If $z_\theta \neq 1$ one has $d_i^{2n+1} = -1$. Then $(d_1 d_2 \cdots d_{2n+1})^{2n+1} = -1$. But $d_1 d_2 \cdots d_{2n+1} = 1$. Contradiction. Hence $z_{\theta,\psi} = 1$.

2. $G = \mathrm{SO}_{2n+1}$. In this case the center of G is trivial.
3. G is a similitude group. Assume $D_\psi = \mathrm{diag}(d_1, \ldots, d_{2n+1})$. Then $D_\psi \theta^{D_\psi}(D_\psi) = D_\psi^2 = z_{\theta,\psi}^{-1} = \mathrm{diag}(z, \ldots, z)$. Since there is no condition on the determinant, one has $z_{\theta,\psi}^{-1} = z_1 \theta^{D_\psi}(z_1)$, where $z_1 = \mathrm{diag}(d_1, \ldots, d_1)$.

Proof of (ii): 1. $G = \mathrm{GL}_{2n+1}, \mathrm{SL}_{2n+1}$. In this case $\theta^{D_\psi}(g) = n_{w_\Delta} {}^t g^{-1} n_{w_\Delta}-1$. Assume $D_\psi = \mathrm{diag}(d_1, \ldots, d_{2n+1})$. One has

$$\begin{aligned} D_\psi \theta^{D_\psi}(D_\psi) &= \mathrm{diag}(d_1, \ldots, d_{n+1}, \ldots, d_{2n+1}) \times \mathrm{diag}({d_{2n+1}}^{-1}, \ldots, d_{n+1}^{-1}, \ldots, d_1^{-1}) \\ &= \mathrm{diag}(\ldots, 1; \ldots) = {z_{\theta,\psi}}^{-1}. \end{aligned}$$

Hence $z_{\theta,\psi} = 1$.

2. $G = \mathrm{SO}_{2n}, \mathrm{GSO}_{2n}$. In this case $\theta^{D_\psi}(g) = B {}^t g^{-1} B^{-1}$, where B is some element in G. For $G = \mathrm{SO}_{2n}$ assume $D_\psi = \mathrm{diag}(d_1, \ldots, d_n, d_n^{-1}, \ldots, d_1^{-1})$. Then $D_\psi \theta^{D_\psi}(D_\psi) = \mathrm{diag}(\ldots, 1, 1, \ldots) = z_{\theta,\psi}$. Hence $z_{\theta,\psi} = 1$. The same argument works for $G = \mathrm{GSO}_{2n}$.

(b) First prove the statement when θ is such that

$$\theta^{D_\psi}(g) = n_{w_\Delta} {}^t g^{-1} n_{w_\Delta}^{-1}.$$

Assume there is an admissible coset represented by tn_w. Then from the proof of Proposition 2 one has

$$\begin{gathered} \tilde{t} n_{w_\Delta} \theta^{D_\psi}(\tilde{t}) {n_{w_\Delta}}^{-1} n_w \theta^{D_\psi}(n_w) = z_{\theta,\psi} \Leftrightarrow \\ n_{w_\Delta} n_{w_J} n_{w_\Delta} {}^t n_{w_\Delta}^{-1} {}^t n_{w_J}^{-1} {n_{w_\Delta}}^{-1} = z_{\theta,\psi} \Leftrightarrow z_{\theta,\psi} = 1. \end{gathered}$$

Contradiction. Thus we have proved the statement for groups $\mathrm{Sp}_{2n}, \mathrm{SO}_{4n}$ if θ acts trivially on the Dynkin diagram and for all groups if θ acts nontrivially on it. So we must check the statement for the groups $G = \mathrm{SL}_{2n}, \mathrm{SO}_{4n+2}$ and θ acts trivially on the Dynkin diagram. It is possible to do it directly, carefully checking different cases of J. We omit here the details but let us illustrate the point with one example. Consider the group SL_4 and an involution θ on it such that θ acts trivially on the Dynkin diagram of SL_4. Then

$$\theta^{D_\psi} = EgE^{-1}, \quad \text{where } \alpha(E) = -1 \text{ for any } \alpha \in \Delta.$$

Denote the simple roots of SL_4 by α_1, α_2 and α_3. Assume that there is an admissible orbit, say tn_w. Then by Proposition 2, $w = w_0 w_J$ for some $J \in \Delta$,

$\tilde{t}n_w\theta^{D_\psi}(\tilde{t}n_w) = z_{\theta,\psi}$, and $\tilde{t} \in Z_{M_J}$. Our aim is to prove that $z_{\theta,\psi}$ is equivalent to 1. Note that J should be symmetric in a sense that $w_J = w_0 w_J w_0$. Consider for example $J = \{\alpha_1, \alpha_3\}$. Then $\tilde{t} = \mathrm{diag}(t, t, t^{-1}, t^{-1})$. It is easy to check that then $z_{\theta,\psi} = 1$. The case $J = \{\alpha_2\}$ is treated similarly. Note that the type of SL_4 is the same as of SO_6, so we have actually illustrated the point for both cases SL_{2n} and SO_{4n+2}. ∎

11.5. THE CASE θ IS GALOIS. This case is more complicated. We cannot completely characterize all cases when $z_{\theta,\psi} = 1$. Note that when $z_{\theta,\psi} \neq 1$ there are still admissible cosets, but they do not match naturally to admissible double cosets of G'. We emphasize that we can choose ψ such that $z_{\theta,\psi} = 1$ in many important cases, like Base Change and its dual.

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References

[F] Y. Flicker, *On distinguished representations,* Journal fur die Reine und Angewandte Mathematik **418** (1991), 139–172.

[JLR] H. Jacquet, K. F. Lai and S. Rallis, *A trace formula for symmetric spaces,* Duke Mathematical Journal **70** (1993), 305–372.

[JR] H. Jacquet and S. Rallis, *Kloosterman integrals for skew symmetric matrices,* Pacific Journal of Mathematics **154** (1992), 265–290.

[JY1] H. Jacquet and Y. Ye, *Une remarque sur le changement de base quadratic,* Comptes Rendus de l'Académie des Sciences, Paris **311** (1990), 671–676.

[JY2] H. Jacquet and Y. Ye, *Distinguished representations and quadratic base change for* GL_3, Transactions of the American Mathematical Society **348** (1996), 913–940.

[J1] H. Jacquet, *The continuous spectrum of the relative trace formula for* GL_3 *over a quadratic extension,* Israel Journal of Mathematics **89** (1995), 1–59.

[J2] H. Jacquet, *Representations distinguees pour le groupe orthogonal,* Comptes Rendus de l'Académie des Sciences, Paris, série 1 **312** (1991), 957–961.

[J3] H. Jacquet, *Sur un resultat de Waldspurger,* Annales Scientifiques de l'École Normale Supérieure **19** (1986), 185–229.

[S] T. A. Springer, *Some results on algebraic groups with involutions,* Advances Studies in Pure Mathematics **6** (1985), 525–543.